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On Banach Lattices and Spaces Having Local Unconditional Structure, with Applications to Lorentz Function Spaces

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In the present paper an attempt is made to study relationships between Banach lattices and spaces which have local unconditional structure (l.u.st.). (Relevant definitions appear below.) In the case where the Banach lattice is $L_p(\mu)$ for some measure μ and the space with l.u.st. is an $\mathscr{L}_p$-space (cf. [20]), there are known to be good relationships between these structures. We recall in particular the beautiful result of Lindenstrauss–Pelczynski [20] and Lindenstrauss–Rosenthal [21]: A Banach space X which is not isomorphic to a Hilbert space is a $\mathscr{L}_p$-space if and only if X^{**} is isomorphic to a complemented subspace of $L_p(\mu)$ for some measure μ. This theorem suggests that a Banach space X has l.u.st. if and only if X^{**} is isomorphic to a complemented subspace of a Banach lattice. The "only if" part of this assertion is proved in Section 2. Indeed, an easy modification of an argument from [20] yields that X has l.u.st. iff X is isomorphic to a subspace of a Banach lattice L so that L is finitely represented in X in a very nice way (in particular, nice enough to ensure that X^{**} must be complemented in L^{**}). This result also

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allows us to prove the "if" part of the conjecture in some special cases (e.g., if X contains l_∞^n's uniformly for large n's). The general case would follow from an affirmative answer to:

MAIN CONJECTURE. If X is a complemented subspace of a Banach lattice, then X has l.u.st.

If the main conjecture is true, then it follows that X has l.u.st. iff X^* has l.u.st. In Section 2 we prove the weaker result that X has l.u.st. iff X^{**} has l.u.st.

In Section 3 we prove some additional embedding theorems. In particular, we prove that if X is a subspace of a space which has an unconditional basis and l_1 does not embed into X (respectively, X is reflexive), then X embeds into a space with shrinking unconditional basis (respectively, reflexive space with an unconditional basis).

Atomic Banach lattices (i.e., Banach spaces which in the separable case have an unconditional basis) are the most important Banach lattices from the point of view of the modern theory of the geometry of Banach spaces. In Section 4 we investigate the well-known problem whether every Banach space X has a subspace with an unconditional basis. An affirmative answer to this question is given in case X is a subspace of a σ-complete and σ-order continuous Banach lattice. In view of the embedding theorem of Section 2, this yields that if X has l.u.st. and X does not contain l_∞^n's uniformly for all n, then every subspace of X has an unconditional basic sequence. This result is improved on in Section 5 for the Lorentz function spaces $\Lambda(W, p)$. Here it is shown that a subspace of $\Lambda(W, p)$ $(1 \leqslant p < \infty)$ either embeds isomorphically into $L_p[0, 1]$ or contains a complemented subspace isomorphic to l_p.

In general, we follow the notation in [22]. X, Y, Z, etc., represent infinite dimensional Banach spaces; subspaces are assumed infinite dimensional and closed. A space X which is a lattice under $\leqslant$ is called a Banach lattice provided $\| x \| \leqslant \| y \|$ whenever $| x | \leqslant | y |$, where $| x | = \sup(x, -x)$. A sequence (e_n) in X is said to be an unconditional basic sequence provided that there are biorthogonal functionals (e_n^*) in X^* so that $\sum e_n^*(x)\, e_n$ converges unconditionally to x for each x in the closed linear span $[e_n]$ of (e_n). In this case there is a constant K so that $\| \sum \alpha_n e_n \| \leqslant K \| \sum \beta_n e_n \|$ whenever $| \alpha_n | \leqslant | \beta_n |$. The smallest such K is denoted by $U(e_n)$ and is called the unconditional constant of (e_n). One can introduce an equivalent norm on X so that $U(e_n) = 1$; this makes $[e_n]$ into an atomic Banach lattice under the pointwise ordering on the coefficients of the basis vectors (e_n). When $U(e_n) = 1$, the basis (e_n) is called unconditionally monotone.

A Banach lattice L is: σ-complete, provided that if $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x$, then $\sup x_n$ exists; σ-order continuous, provided that if $x_1 \geqslant x_2 \geqslant \cdots \geqslant 0$ and $0 = \inf(x_n)$, then $\| x_n \| \to 0$. It is known [24, 26, 30] that X is σ-complete and σ-order continuous if no subspace (or sublattice) of X is isomorphic (i.e., linearly homeomorphic) to c_0.

Given isomorphic normed spaces E and F, $d(E, F)$ denotes $\inf\{\|T\| \cdot \|T^{-1}\|\}$, where the inf is taken over all invertible operators (= linear operators) from E onto F. X has local unconditional structure (l.u.st.) provided $X = \bigcup_\alpha E_\alpha$ where the E_α's are finite dimensional subspaces of X forming an increasing net when directed by inclusion, and E_α has a basis $(e_i^\alpha)_{i=1}^{n(\alpha)}$ for which $\sup_\alpha U(e_i^\alpha)_{i=1}^{n(\alpha)} = K < \infty$. $LU(X)$ is the infimum of all such K, as (E_α) ranges over all such decompositions of X. The concept of l.u.st. was introduced in [8].

If X is a σ-complete Banach lattice, then it is easy to show that for any finite dimensional subspace E of X and $\epsilon > 0$, there are disjointly supported vectors $(x_i)_{i=1}^n$ in X (i.e., $|x_i| \wedge |x_j| = 0$ for $i \neq j$) and an operator $T: E \to [x_i]_{i=1}^n$ so that $\|T - I_E\| < \epsilon$. Since $U(x_i) = 1$, a perturbation argument yields that $LU(X) = 1$. Since Y^{**} is a σ-complete Banach lattice for any Banach lattice Y, it follows from the principle of local reflexitivity [21] that $LU(Y) = 1$ for every Banach lattice Y.

X is said to be finitely representable in Y provided for all finite dimensional subspaces E of X, $\inf\{d(E, F): F \subseteq Y\} = 1$. X is super-reflexive (cf. [14]) provided every space finitely represented in X is reflexive. Enflo [9] proved that if X is super-reflexive, then X can be given an equivalent uniformly convex norm.

X is said to contain l_p^n uniformly for all n provided there is a sequence (E_n) of subspaces of X for which $\sup_n d(E_n, l_p^n) < \infty$. If the E_n's can be chosen so that there are projections P_n from X onto E_n with $\sup_n \|P_n\| < \infty$, then X is said to contain uniformly complemented l_p^n's for all n.

A basic sequence (e_n) is said to be λ-equivalent to a basic sequence (f_n) provided the map $e_n \to f_n$ extends to an isomorphism T from the closed linear span $[e_n]$ of (e_n) onto $[f_n]$ so that $\max\{1, \|T\|\} \max\{1, \|T^{-1}\|\} \leqslant \lambda$.

2. *The Embedding Results and Related Topics*

The proof that a Banach space with l.u.st. embeds in a good way into a Banach lattice is modelled on the proof of Theorem 7.1 in [20]. One could also give a proof using ultraproduct techniques (cf. [6]).

THEOREM 2.1. *A Banach space X has* l.u.st. *if and only if there are a Banach lattice L, $\lambda < \infty$, and a subspace Y of L which is isomorphic to X and satisfies the following: Given any finite dimensional subspace E of L there is an operator $T = T_E : E \to Y$ for which $T|_{E \cap Y} = I_{E \cap Y}$ and $\|T\| \cdot \|T^{-1}\| \leqslant \lambda$.*

Proof. The if part is very easy. Indeed, if F is a finite dimensional subspace of Y then since $LU(L) = 1$ we know that there is a finite dimensional subspace

E of L with $F \subseteq E$ and $U(E) \leqslant 2$. Then TE is a subspace of Y which contains F and $U(TE) \leqslant 2\lambda$. Hence Y has l.u.st., whence X has l.u.st.

To prove the only if part, write $X = \bigcup E_\alpha$, where the E_α's are finite dimensional, directed by inclusion, and each E_α has a basis $(e_i^\alpha)_{i=1}^{n(\alpha)}$ with $U(e_i^\alpha) < \lambda$ for each α and for some λ. Let (f_i^α) be Hahn–Banach extensions of the functionals on E_α biorthogonal to (e_i^α) to elements of X^*, and assume by normalization that $\| f_i^\alpha \| = 1$. Since $U(e_i^\alpha) < \lambda$, there is a new norm $| \cdot |_\alpha$ on E_α with respect to which (e_i^α) is unconditionally monotone and which satisfies $\| x \| \leqslant | x |_\alpha \leqslant \lambda \| x \|$ for $x \in E_\alpha$.

Let Z be the collection of all real valued functions on the unit ball B_{X^*} of X^*. For each α, define $T_\alpha : Z \to E_\alpha$ by $T_\alpha g = \sum_{i=1}^{n(\alpha)} g(f_i^\alpha) e_i^\alpha$. By passing to a subnet of (T_α) (recall that the E_α's are directed by inclusion) we may assume that $||| g ||| = \lim_\alpha | T_\alpha g |_\alpha$ exists in the extended reals for each $g \in Z$. Let $L = \{ g \in Z: ||| g ||| < \infty \}$. It is routine to verify that L is a Banach lattice under the pointwise order (of course, we identify g, h in L if $||| g - h ||| = 0$). Indeed, if $| g | \leqslant | h |$ then $| T_\alpha g |_\alpha \leqslant | T_\alpha h |_\alpha$ for each α since (e_i^α) is unconditionally monotone with respect to $| \cdot |_\alpha$; thus $||| g ||| \leqslant ||| h |||$. We omit the routine proof that L is complete.

For $x \in X$ let $Jx \in Z$ be defined by $Jx(x^*) = x^*(x) (x^* \in B_{X^*})$. If $x \in E_\alpha$ then for $\beta > \alpha$, $\| x \| \leqslant | T_\beta Jx |_\beta \leqslant \lambda \| x \|$, hence $\| x \| \leqslant ||| Jx ||| \leqslant \lambda \| x \|$ for $x \in X$, whence J is an isomorphism of X into L.

Finally suppose E is a finite dimensional subspace of L. Then for large α, $T_{\alpha|E}$ is almost an isometry and $T_\alpha Jx = x$ for $x \in E \cap JX$ as long as $E_\alpha \supseteq J^{-1}(E \cap JX)$. Thus $Y = JX$ satisfies the desired condition (set $T_E = JT_\alpha$ for α sufficiently large).

Corollary 2.2. *Every Banach space X with* l.u.st. *is isomorphic to a subspace Y of a Banach lattice L such that* (a) *L is finitely representable in Y;* (b) *There exists a projection P on L^* whose range is isomorphic to Y^* and for which $(I - P) L^* = Y^\perp$. Consequently, if X is complemented in X^{**} then X is isomorphic to a complemented subspace of L.*

Proof. Direct the finite dimensional subspaces of L by inclusion and consider the net $\{T_E : E$ is a finite dimensional subspace of $L\}$. By passing to a subnet of this net we may assume that $f(T_E)x$ converges for each $f \in L^*$ and $x \in L$, say, $f(T_E) x \to (Pf)(x)$. One easily verifies that P is linear, $Pf \in L^*$ for $f \in L^*$, P is bounded, and kernel $P = Y^\perp$. P is a projection because $(PPf)(x) = \lim_E (Pf) T_E x = \lim_E \lim_F f(T_F T_E x)$. But for fixed E, $\lim_F f(T_F T_E) x = f(T_E x)$ (since $T_E E \subseteq F \cap Y$ eventually and $T_{F|F \cap Y} = I_{F \cap Y}$) so $(PPf)(x) = Pf(x)$ and P is a projection.

It is a well-known application of the Hahn–Banach theorem that the conditions on P imply that PL^* is isomorphic to Y^*. Finally, if Q is a projection from Y^{**} onto Y, then (identifying $Y^{\perp\perp} \subseteq L^{**}$ with Y^{**}) we have that $QP^*_{|L}$ is a projection of L onto Y.

Remark 2.3. Gordon and Lewis [12] define local unconditional structure differently from us. A modification of the proofs of Theorem 2.1 and Corollary 2.2 yields that X has local unconditional structure in their sense, if and only if X^{**} is complemented in a Banach lattice.

COROLLARY 2.4. *Suppose X is complemented in a Banach lattice L and X contains $l_\infty{}^n$ uniformly for all n. Then has* l.u.st.

Proof. Given a finite dimensional subspace E of X an integer n, there is a subspace F of X with $\| e + f \| \geqslant \frac{1}{2} \| e \|$ $(e \in E, f \in F)$ and $d(F, l_\infty{}^n) \leqslant 2$ (cf., e.g. Lemma 4.1 of [11]). Now if $L = X \oplus Y$ and G is a finite dimensional subspace of L, one can pick finite dimensional $E \subseteq X$, $H \subseteq Y$ with $G \subseteq E + H$. By the remark at the beginning of the proof and the universality of $l_\infty{}^n$ we can choose a subspace W of X, an isomorphism τ from H into W with $\| \tau \| = 1$, $\| \tau^{-1} \| \leqslant 2$, so that $\| e + w \| \geqslant \frac{1}{2} \| e \|$ for $e \in E$ and $w \in W$. Define $T: E + H \to X$ by $T(e + h) = e + \tau h$. Then $T_{|(E+H)\cap X} = I_E$ and $\| T \| \cdot \| T^{-1} \|$ is bounded independently of E and H. Thus by the easy implication of Theorem 2.1, X has l.u.st.

Note that a particular case of Corollary 2.4 is that $X \oplus c_0$ has l.u.st. if X is complemented in a Banach lattice.

Remark 2.5. The proof of Corollary 2.4 is very similar to Lindenstrauss and Rosenthal's proof [21] that a complemented subspace of L_p which is not isomorphic to a Hilbert space is a $\mathscr{L}_p$ space.

We wish to make some further comments on the problem whether every complemented subspace of a Banach lattice has l.u.st. The natural approach to the problem is the following: Given a complemented subspace X of a lattice L, renorm L with a new lattice norm so that the new norm is equivalent the old norm on X and L under the new norm is finitely represented in X in a good way. Although we are unable to find such a new norm, this approach does give some information:

PROPOSITION 2.6. *Assume X is complemented in a σ-complete lattice $(L, \| \cdot \|)$.* (i) *If X does not contain $l_\infty{}^n$ uniformly for large n, then X is complemented in a lattice which does not contain $l_\infty{}^n$ for large n.* (ii) *If X is super-reflexive, then X is complemented in a super-reflexive lattice.*

Proof. Let P be a bounded projection of L onto X. In case (i), define a seminorm $|\!|\!| \cdot |\!|\!|$ on L by $|\!|\!| y |\!|\!| = \sup\{\| Pz \|: | z | \leqslant | y |\}$. Since $\{0\} \cup \{y \in L: |\!|\!| y |\!|\!| > 0\}$ is a σ-complete sublattice of L containing X, we may and do assume that $|\!|\!| \cdot |\!|\!|$ is a norm on L.

For $x \in X$, $\| x \| \leqslant ||| x ||| \leqslant \| P \| \cdot \| x \|$, so $||| \cdot |||$ is equivalent to $\| \cdot \|$ on X. Obviously if $y, z \in L$ with $| y | \leqslant | z |$, then $||| y ||| \leqslant ||| z |||$, hence the completion of $(L, ||| \cdot |||)$ is a Banach lattice. We next observe that P is continuous in the $||| \cdot |||$ norm. Indeed, for $y \in L$, $Py \in X$, so $||| Py ||| \leqslant \| P \| \cdot \| Py \| \leqslant \| P \| \cdot ||| y |||$, hence $||| P ||| \leqslant \| P \|$.

To complete the proof we must show that $(L, ||| \cdot |||)$ does not contain l_∞^n uniformly for all n. Now an extension by Maurey [25] of a theorem due to Rosenthal [28] implies that there is $p < \infty$ and a positive constant λ so that $\max_{\epsilon_i = \pm 1} \| \sum_{i=1}^n \epsilon_i x_i \| \geqslant \lambda (\sum \| x_i \|^p)^{1/p}$ for every sequence $(x_i)_{i=1}^n$ in X. Suppose that $(y_1, y_2, \ldots, y_k)$ are disjoint vectors in L, and $| y_i | \geqslant | z_i |$ for $1 \leqslant i \leqslant k$.

For any choice $(\epsilon_i)_{i=1}^k$ of signs, we have from the definition of $||| \cdot |||$ that

$$\left\| P\left(\sum_{i=1}^{k} \epsilon_i z_i\right)\right\| \leqslant \left|\left|\left| \sum_{i=1}^{k} y_i \right|\right|\right| .$$

But then

$$\lambda \left(\sum_{i=1}^{k} \| P z_i \|^p\right)^{1/p} \leqslant \left|\left|\left| \sum_{i=1}^{k} y_i \right|\right|\right| ,$$

hence

$$\lambda \left(\sum_{i} ||| y_i |||^p\right)^{1/p} \leqslant \left|\left|\left| \sum_{i=1}^{k} y_i \right|\right|\right| .$$

This shows that there is a k so that no disjointly supported sequence of vectors in $(L, ||| \cdot |||)$ is 2-equivalent to the unit vector basis for l_∞^k. By the results of [15], there is an n so that if E is a subspace of $(L, ||| \cdot |||)$ and E is contained in a subspace of L which is spanned by a disjointly supported sequence of vectors, then $d(E, l_\infty^n) \geqslant 2$. Now suppose E is a finite dimensional subspace of L. Since $(L, \| \cdot \|)$ is σ-complete, the remarks in the introduction yield that there is for each $\epsilon > 0$ an operator T from E into a subspace of L spanned by a disjointly supported sequence so that $\| Te - e \| < \epsilon \| e \| \leqslant \epsilon K ||| e |||$ each $e \in E$. (K is just the norm of the identity I: $(E, ||| \; |||) \to (E, \| \cdot \|)$.) That is, $||| T - I_E ||| < \epsilon K$. Letting $\epsilon \to 0$, we have $d(E, l_\infty^n) \geqslant 2$.

We turn now to the proof of (ii). Since X^* is complemented in the lattice L^*, X^* is complemented in another lattice M which does not contain l_∞^n uniformly for all n, hence $X = X^{**}$ is complemented in a σ-complete lattice (namely, M^*, which does not contain uniformly complemented l_1^n's for large n. Thus without loss of generality, we may assume that L does not contain uniformly complemented l_1^n's for large n. Now define a new norm $||| \cdot |||$ on L as in the first part of the proof. Then $(L, ||| \cdot |||)$ does not contain l_∞^n's, uniformly for large n. Now it is well-known (cf. e.g., the proof of Proposition II.6 in [15]) that if $(y_i)_{i=1}^n$ is a disjointly supported sequence in a lattice which is λ-equivalent to the unit vector basis for l_1^n, then

$[y_i]$ is λ-complemented in the lattice. Hence there is an integer n so that no length n disjointly supported sequence in $(L, \|\cdot\|)$ is 2-equivalent to the unit vector basis for $l_1{}^n$. By Lemma III.1 of [15] we have that there is $q > 1$ so that $\|\sum y_i\| \leqslant 2(\sum \|y_i\|^q)^{1/q}$ for any disjointly supported sequence (y_i) in L. It then follows from the decomposition lemma for lattices * and the definition of $|||\cdot|||$ that $|||\sum y_i||| \leqslant 2(\sum |||y_i|||^q)^{1/q}$ for any disjointly supported sequence (y_i) in L. Summarizing, we have that there is an m so that if $(y_i)_{i=1}^m$ is disjointly supported in $(L, |||\cdot|||)$, then $(y_i)_{i=1}^m$ is not 2-equivalent to the unit vector basis for $l_1{}^m$ or $l_\infty{}^m$. As in the proof of part (i), every finite dimensional subspace of $(L, |||\cdot|||)$ is for each $\epsilon > 0$ ϵ-isometric to a subspace of $(L, |||\cdot|||)$ spanned by disjointly supported vectors so that the main result of [15] implies that the completion of $(L, |||\cdot|||)$ is super-reflexive.

Remark 2.7. The proof of (i) of Proposition 2.6 shows that if X is complemented in a space with unconditional basis and X does not contain $l_\infty{}^n$ uniformly for large n, then X is complemented in a space with unconditional basis which does not contain $l_\infty{}^n$ uniformly for large n. Part (ii) of Proposition 2.6 also generalizes to the unconditional basis case, but more work is necessary because the conjugate space to a space with unconditional basis need not have an unconditional basis. One need first to show that if X is complemented in a space with unconditional basis, and l_1 does not isomorphically embed into X, then X is complemented in a space with shrinking unconditional basis. This assertion is proved in Theorem 3.3.

Remark 2.8. The hypothesis in Proposition 2.6 that L be σ-complete is superfluous. In the non σ-complete case renorm L in the same way. The proof of Proposition 2.6 shows that no disjointly supported sequence in $(L, |||\cdot|||)$ is equivalent to the unit vector basis of c_0. Thus the completion of $(L, |||\cdot|||)$ is σ-complete by the remarks in the introduction and, by [30], it is complemented in its second dual. Consequently, X is also complemented in its second dual X^{**}.

We do not know whether it is true that if X has l.u.st. then X^* has l.u.st. (Of course, this would follow if our main conjecture has an affirmative answer). However, we can show that X^{**} has l.u.st. whenever X does. (The converse is obvious from the principle of local reflexivity, [21].)

We need a preliminary lemma which says that every Banach space is locally complemented in its bidual. The motivation for the lemma is the proof of the well-known fact that X^* is norm one complemented in X^{**}.[1]

LEMMA 2.9. *Suppose* $X \subseteq Y \subseteq X^{**}$ *with* $\dim Y/X < \infty$. *Then for each* $\epsilon > 0$, X *is* $3 + \epsilon$ – *complemented in* Y.

[1] i.e., if $0 \leqslant x \leqslant y + z$ with $y, z \geqslant 0$ then $x = x_1 + x_2$ with $0 \leqslant x_1 \leqslant y$ and $0 \leqslant x_2 \leqslant z$.

Proof. We can write $Y^* = X^\perp + JX^*$, where $J: X^* \to Y^*$ is the isometry defined by $(Jx^*)\, y = y(x^*)$. The projection Q from Y^* onto JX^* perpendicular to $X^\perp$ has norm 1, so $\| I_{Y^*} - Q \| \leqslant 2$. Since $X^\perp$ is finite dimensional, there is by Corollary 3.2 of [17] for each $\epsilon > 0$ a projection P on Y with $\| P \| \leqslant 2 + \epsilon$ and $P^* Y^* = X^\perp$. Obviously $(I - P)\, Y = X$.

THEOREM 2.10. *If* Y *has* l.u.st., *then* Y^{**} *has* l.u.st.

Proof. We can assume that Y is a subspace of a lattice L and there is a $\lambda < \infty$ so that Y, L, λ satisfy the conclusion of Theorem 2.1.

We wish to show that Y^{**} in the lattice L^{**} satisfies the condition in Theorem 2.1. So let H be a finite dimensional subspace of L^{**}, and let $\epsilon > 0$ be arbitrary. By replacing H with a larger finite dimensional subspace of L^{**}, we can assume by Lemma 2.9 and the fact that Y^{**} is λ-complemented in L that $H = A + B + C$ where A, B, C are finite dimensional subspaces of Y, Y^{**} and L^{**}, respectively, satisfying

(i) $\| y + b \| > (3 + \epsilon)^{-1} \| y \| \ (y \in Y, b \in B)$,

(ii) $\| y^{**} + c \| > (\lambda + \epsilon)^{-1} \| y^{**} \| \ (y^{**} \in Y^{**}, c \in C)$.

By (ii) and local reflexivity in L^{**} (cf. [21]), there is an operator $\tau: C \to L$ so that $\| \tau \| \| \tau^{-1} \| \leqslant 1 + \epsilon$ and

(iii) $\| a + \tau c \| > (\lambda + \epsilon)^{-1} \| a \| \ (a \in A, c \in C)$.

We have from (ii) and the condition on Y, L that there is an operator $S: A + \tau C \to Y$ with $\| S \| \leqslant \lambda + \epsilon$, $\| S^{-1} \| = 1$, and $S_{|A} = I_A$.

Consider the operator $T: H \to Y^*$ defined by $T(a + b + c) = a + b + S\tau c$ $(a \in A, b \in B, c \in C)$. Obviously $T_{|H \cap Y^{**}} = I_{H \cap Y^{**}}$. Also it is easy to check that $\| T \| \cdot \| T^{-1} \| < M$ for some constant $M = M(\lambda, \epsilon)$. To see this, note that from (i) and (ii) we have a positive constant K (with $K \geqslant (3 + \epsilon)^{-1}(\lambda + 1 + \epsilon)^{-1}$) so that

(iv) $K \max(\| a \|, \| b \|, \| c \|) \leqslant \| a + b + c \| \leqslant 3 \max(\| a \|, \| b \|, \| c \|)$ $(a \in A, b \in B, c \in C)$.

On the other hand, we have from (i) that

$$\| a + b + S\tau c \| \geqslant (4 + \epsilon)^{-1} \| b \| \quad \text{and} \quad \| a + b + S\tau c \| \geqslant (3 + \epsilon)^{-1} \| a + S\tau c \|.$$

But from (iii) and the condition on S it follows that $\| a + S\tau c \| \geqslant (4 + \epsilon)^{-1} (\lambda + \epsilon)^{-1} \max(\| a \|, \| S\tau c \|)$. Since $\| S\tau \| \| (S\tau)^{-1} \| \leqslant (\lambda + \epsilon)(1 + \epsilon)$, we get a positive constant K_1 (actually $K_1 \geqslant (4 + \epsilon)^{-1} (\lambda + \epsilon)^{-2} (1 + \epsilon)^{-1}$) so that

(v) $K_1 \max(\| a \|, \| b \|, \| c \|) \leqslant \| a + b + S\tau c \| \leqslant 3(\lambda + \epsilon)(1 + \epsilon) \max(\| a \|, \| b \|, \| c \|)$.

Of course, (iv) and (v) yield that $\|T\| \|T^{-1}\| < M$ for some constant $M = M(K, K_1, \lambda)$ (in fact, $M \leqslant 9(\lambda + \epsilon)(1 + \epsilon) K^{-1}K_1^{-1}$).

3. Other Embedding Theorems

Our next embedding result is an immediate application of the factorization theorem of [7] and a result of Abramovich's [1]. The unconditional basis case answers in the affirmative a question of Bessaga and Pelczynski [4].

THEOREM 3.1. *Assume that X has an unconditional basis (respectively, X is a σ-complete and σ-order continuous Banach lattice) and Z is a reflexive subspace of X. Then Z isomorphically embeds into a reflexive space Y so that Y has an unconditional basis (respectively, Y is a Banach lattice).*

Proof. Let $W_0 = \{x \in X: \exists z \in Z \text{ with } \|z\| \leqslant 1 \text{ and } |x| \leqslant |z|\}$. By the main result of [1], W_0 is a weakly compact set, hence so is $W = \overline{\text{conv}}\, W_0$. Obviously if $x \in X$ and $|x| \leqslant |w|$ for some $x \in W$, then $x \in W$. Thus if we apply the factorization process of Lemma 1 in [7] to W, we get a space Y which has an unconditional basis (respectively, which is a Banach lattice). Y is reflexive since W is weakly compact and Z isomorphically embeds into Y since W contains the unit ball of Z.

We wish to show next that if Z does not contain an isomorph of l_1 and Z is a subspace of a Banach space which has an unconditional basis, then Z is isomorphic to a subspace of a Banach space which has a shrinking unconditional basis. In preparation for this, we need a variation of the result of Abramovich used in the proof of Theorem 3.1. Let us recall that a subset A of Z is said to be *weak* sequentially compact* provided thate very sequence in A has a weakly Cauchy subsequence. It is well-known that if V is a bounded subset of a space X which has an unconditional basis and V is not weak* sequentially compact, then V contains a sequence which is equivalent to the unit vector basis of l_1. (Recently Rosenthal [29] has shown that the condition that X have an unconditional basis is superfluous.)

LEMMA 3.2. *If X is a space with an unconditional basis and V is a weak* sequentially compact subset of X, then so is the closed convex hull W of the set $V^1 = \{x \in X: |x| \leqslant |v| \text{ for some } v \in V\}$.*

Proof. If W is not weak* sequentially compact, then by the above remarks there is a sequence (w_n) in W which is equivalent to the unit vector basis of l_1. Clearly we may assume that the supports of the w_n are finite and disjoint.

Let $F \in X^*$ be such that $F(w_n) > 1$ for all n. Since $F(w_n) \leqslant \sup\{F(x): x \in V'$ and $\text{supp}\, x \subseteq \text{supp}\, w_n\}$, there are for each n a $v_n \in V$ and a function λ_n defined on the set of indices of the basis so that $|\lambda_n| \leqslant 1$, $\text{supp}\, \lambda_n \subseteq \text{supp}\, w_n$, and $1 = F(\lambda_n \cdot V_n) = (\lambda_n \cdot F)(v_n)$. (The dot denotes coordinatewise multiplication.) $(\lambda_n \cdot F)$ is equivalent to the unit vector basis of c_0 since $\| \sum_{n \in A} \lambda_n \cdot F \| \leqslant \| F \|$ for any set A of indices and $\inf \| \lambda_n \cdot F \| > 0$. In particular, the series $\sum_{n \in A} \lambda_n \cdot F$ converges weak* in X^* for any set A of indices.

By hypothesis on V, there is a weakly Cauchy subsequence (V_{n_i}) of (v_n). This is absurd, since $G = \sum_{i=1}^{\infty} (-1)^i \lambda_{n_i} F_{n_i}$ satisfies $G(v_{n_i}) = (-1)^i$.

THEOREM 3.3. *Suppose T: $Z \to X$ is an operator, $T(Ball\, Z)$ is weak* sequentially compact in X, and X has an unconditional basis. Then there is a space Y with shrinking unconditional basis and operators A: $Z \to Y$ and B: $Y \to X$ which satisfy $BA = T$. Hence if T is an isomorphic embedding, then so is A, and if also TZ is complemented in X then AZ is complemented in Y. Finally, if T is weakly compact, then Y is reflexive.*

Proof. We again use the factorization technique in Lemma 1 of [7] and follow the notation used in [7]. As in Theorem 3.1, let W be the closed convex hull of the set $\{x \in X: | x | \leqslant | Tz | \text{ for some } z \in Ball(Z)\}$. The factorization process of Lemma 1 yields the space Y. Y has an unconditional basis by part (x) of Lemma 1 which is shrinking by part (ix). The complementation follows from (viii) of this lemma. By Abramovich's result [1] W is weakly compact if T is a weakly compact operator, so reflexivity of Y follows from part (iv) of the lemma.

Remark 3.4. We ask whether there is a dual version of Theorem 3.2, namely, that if c_0 does not isomorphically embed into X and X is a subspace of a space with unconditional basis, then is X a subspace of a space which has a boundedly complete unconditional basis.

4. SUBSPACES OF LATTICES WITH UNCONDITIONAL BASES

The main theorem we prove in this section is Theorem 4.1.

THEOREM 4.1. *Let L be a σ-complete and σ-order continuous Banach lattice. Suppose X is a subspace of L. Either X is isomorphic to a subspace of $L_1(\mu)$ for some measure μ or there is a sequence (x_i) of unit vectors in X and a disjointly supported sequence (e_i) in L with $\| x_i - e_i \| \to 0$. Consequently, every subspace of L contains an unconditional basic sequence.*

Proof. If every separable subspace of X is isomorphic to a subspace of $L_1(\nu)$ for some measure ν, then X embeds isomorphically into $L_1(\mu)$ for some measure μ (cf. Proposition 7.1 in [20]). Thus we may assume that X is separable. But then X is contained in a separable σ-complete and σ-order continuous sublattice of L, so we may assume L is separable.

It has been shown in [30] that there is a measure space $(\mu, S, \sum)$, a linear (not necessarily closed) sublattice Y of $L_1(\mu)$ and a norm $\|\cdot\|$ on Y so that $(L, \|\cdot\|)$ is isometric and lattice isomorphic to $(Y, \|\cdot\|)$ and $\int |y|\, d\mu \leqslant \|y\|$ for all $y \in Y$. (Of course, the ordering on $L_1(\mu)$ and Y is pointwise a.e. with respect to μ.) For the sake of completeness we reproduce here a neat proof of this assertion due to Meyer–Nieberg [27]: Since L is separable there is a strictly positive functional $x^* \in L^*$ of norm one (i.e., $x^*(x) > 0$ for each $0 < x \in L$). Define $|||\cdot|||$ on L by $|||x||| = x^*(|x|)$. It is clear that the completion of $(L, |||\cdot|||)$ is an abstract L-space and thus is isometric to $L_1(\mu)$ for some measure μ by the Kakutani representation theorem [19].

We also use a less obvious fact about this construction (Cf. Lemma I.1 of [17]): Y is an order ideal in $L_1(\mu)$; i.e., $y \in Y$, $z \in L_1(\mu)$, and $|z| \leqslant |y|$ imply that $z \in Y$.

From now on we regard L as an order ideal in $L_1(\mu)$ and assume $\int |y|\, d\mu \leqslant \|y\|$ for $y \in L$. Since L is dense in $L_1(\mu)$, there is a norm one vector $f \in L$ with $f(t) > 0$ for almost all t. Thus by mapping $L_1(\mu)$ into $L_1(f\, d\mu)$ (by $y \to y/f$), we can assume without loss of generality that $1 \in Y$ and in fact $\|1\| = 1$. We now use the well known technique of Kadec–Pelczynski [18]. For $y \in L$ and $\epsilon > 0$, let $A(y, \epsilon) = \{t \in S: |y(t)| \geqslant \epsilon \|y\|\}$. Let $M(\epsilon) = \{y \in L: \mu(A(y, \epsilon)) \geqslant \epsilon\}$. Observe that if $y \in M(\epsilon)$, then $\int |y|\, d\mu \geqslant \int_{A(y,\epsilon)} |y|\, d\mu \geqslant \epsilon^2 \|y\|$. Thus if there is an $\epsilon > 0$ so that $X \subseteq M(\epsilon)$, then X is isomorphic to a subspace of $L_1(\mu)$.

In the other case, we can pick a sequence (y_n) in X with $\|y_n\| = 1$ and $y_n \notin M(2^{-n})$. For $m > n$, let $A_{n,m} = A(y_n, 2^{-n}) \setminus (\bigcup_{k=m}^{\infty} A(y_k, 2^{-k}))$. For fixed n, $\mu(A_{n,m}) \to \mu[A(y_n, 2^{-n})]$ as $m \to \infty$, hence $\|\chi_{A_{n,m}} y_n - \chi_{A(y_n, 2^{-n})} y_n\| \to 0$ as $m \to \infty$ by the order continuity of the norm. (Note that $\chi_{A_{n,m}} y_n$ and $\chi_{A(y_n, 2^{-n})} y_n$ are in Y because Y is an order ideal in $L_1(\mu)$). Thus we can pass to a subsequence (y_{n_i}) of (y_n) and find a pairwise disjoint sequence (A_i) of μ measurable sets with $A_i \subseteq A(y_{n_i}, 2^{-n_i})$ so that

$$\|\chi_{A_i} y_{n_i} - \chi_{A(y_{n_i}, 2^{-n_i})} y_{n_i}\| < 2^{-i}.$$

But then

$$\|y_{n_i} - \chi_{A_i} y_{n_i}\| < 2^{-i} + 2^{-n_i} \|y_{n_i} - \chi_{A(y_{n_i}, 2^{-n_i})}\| \leqslant 2^{-i} + 2^{-n_i} \|1\| \leqslant 2^{-i+1}.$$

Therefore $x_i = y_{n_i}$, $e_i = \chi_{A_i} y_{n_i}$ have the desired properties.

For the final statement, we use the deep result of Rosenthal [28] that every subspace of $L_1(\mu)$ contains an unconditional basic sequence in the case where the subspace X of L embeds in $L_1(\mu)$, and a standard stability result otherwise.

As an immediate corollary of Theorems 4.1 and 2.1 we have

COROLLARY 4.2. *If X has* l.u.st. *and X does not contain l_∞^n uniformly for large n, then every subspace of X contains an unconditional basic sequence.*

Proof. By Theorem 2.1, X is isomorphic to a subspace of a Banach lattice which does not contain l_∞^n uniformly for large n; in particular, c_0 is not isomorphic to a subspace of L. Therefore, as mentioned in the introduction, L is σ-complete and σ-order continuous, so Theorem 4.1 applies.

Remark 4.3. If X is a complemented subspace of a reflexive lattice L, and X is not isomorphic to a Hilbert space, then at least one of the spaces X or X^* contains a basic sequence equivalent to a disjointly supported sequence in L (respectively, L^*). Indeed, since X is complemented in L, L^* contains X^* isomorphically, and both L and L^* are σ-complete and σ-order continuous, so if the desired conclusion is false both X and X^* are isomorphic to subspaces of $L_1(\mu)$ for some measure μ. But then by a result of Grothendieck [13, p. 66] (cf. also [20]), X is isomorphic to a Hilbert space.

Remark 4.4. It may be that if X is complemented in a reflexive lattice L, then either X is isomorphic to a Hilbert space or X contains a basic sequence which is equivalent to a disjointly supported sequence in L. We can prove this in the case where L is uniformly convex and X is norm one complemented in L.

5. SUBSPACES OF LORENTZ FUNCTION SPACES

Given a nonincreasing function W on $(0, 1]$ with $\int_0^1 W(t)\,dt = 1$; $W \notin L_\infty[0, 1]$; $W(1) > 0$ and $1 \leqslant p < \infty$, let $\Lambda(W, p)$ be the Lorentz function space of all measurable functions f on $[0, 1]$ for which $\|f\| = (\int_0^1 f^*(t)^p\, W(t)\,dt)^{1/p} < \infty$. Here f^* is the decreasing rearrangement of $|f|$. $\Lambda(W, p)$ is a σ-complete and σ-order continuous Banach lattice which is reflexive if and only if $p > 1$ (cf. [23]).

For our purpose there is a more convenient way of computing the norm in $\Lambda(W, p)$. Note that for $f \in \Lambda(W, p)$, $\|f\| = \sup(\int_0^1 |f(\tau t)|^p\, W(t)\,dt)^{1/p}$ where the sup is taken over all measure preserving automorphisms τ from $[0, 1]$ onto $[0, 1]$.

The first result we prove concerning the spaces $\Lambda(W, p)$ is

THEOREM 5.1. *Let (f_n) be a disjointly supported sequence of norm one vectors in $\Lambda(W, p)$ and let $\epsilon > 0$. Then (f_n) has a subsequence which is $(1 - \epsilon)^{-1}$ equivalent to the unit vector basis for l_p and which spans a $(1 - \epsilon)^{-1}$-complemented subspace of $\Lambda(W, p)$.*

Proof. For A a measurable subset of $[0, 1]$, $|A|$ denotes the Lebesgue measure of A. Let $A_n = \operatorname{supp} f_n$ and choose $\tau_n : [0, |A_n|] \to A_n$ measure preserving so that $\int_0^1 |f_n(\tau_n(t))|^p W(t)\, dt = 1$. Choose $\epsilon_n > 0$ so that

$$\int_{\epsilon_n}^{|A_n|} |f_n(\tau_n(t))|^p W(t)\, dt > (1 - \epsilon)^p$$

and assume by passing to a subsequence of (f_n) that $\epsilon_n > |A_{n+1}|$. We claim that (f_n) is $(1 - \epsilon)^{-1}$-equivalent to the unit vector basis of l_p and $[f_n]$ is $(1 - \epsilon)^{-1}$-complemented in $\Lambda(W, p)$.

Let us first observe that the inequality

$$\left\| \sum a_n f_n \right\|^p \leqslant \sum |a_n|^p, \tag{5.2}$$

for an arbitrary sequence (a_n) of scalars follows from the disjointness of the supports of the f_n's. Indeed, given a measure preserving automorphism τ on $[0, 1]$, we have

$$\int_0^1 \left| \sum a_n f_n(\tau(t)) \right|^p W(t)\, dt$$
$$= \sum |a_n|^p \int_0^1 |f_n(\tau(t))|^p W(t)\, dt \leqslant \sum |a_n|^p \|f_n\|^p \leqslant \sum |a_n|^p.$$

On the other hand, given (a_n), let τ be any measure preserving transformation on $[0, 1]$ for which $\tau = \tau_n$ on $[\epsilon_n, |A_n|]$ for all n. Then

$$\left\| \sum a_n f_n \right\|^p \geqslant \int_0^1 \left| \sum a_n f_n(\tau(t)) \right|^p W(t)\, dt$$
$$\geqslant \sum_n \int_{\epsilon_n}^{|A_n|} |a_n|^p |f_n(\tau_n(t))|^p W(t)\, dt$$
$$\geqslant \sum_n |a_n|^p (1 - \epsilon)^p \geqslant \left(\sum |a_n|^p \right) (1 - \epsilon)^p.$$

Of course, this inequality combines with (5.2) to yield that (f_n) is $(1 - \epsilon)^{-1}$-equivalent to the unit vector basis for l_p.

The proof that $[f_n]$ is $(1 - \epsilon)^{-1}$-complemented in $\Lambda(W, p)$ is based on a proof of Casazza and Lin [5] for an analogous fact concerning Lorentz sequence spaces.

Define functionals F_n on $\Lambda(W, p)$ by

$$F_n(f) = \left(\int_{\epsilon_n}^{|A_n|} f(\tau(t)) |f_n(\tau(t))|^{p-1} \operatorname{sgn} f_n(\tau(t))\, W(t)\, dt \Big/ \int_{\epsilon_n}^{|A_n|} |f_n(\tau(t))|^p W(t)\, dt \right)$$

where τ is the previously defined automorphism of [0, 1]. Obviously

$$F_n(f_n) = 1 \quad \text{and} \quad F_n(f_m) = 0 \quad \text{for} \quad n \neq m.$$

Observe that for $f \in \Lambda(W, p)$,

$$\sum_n |F_n(f)|^p \leqslant \sum \left[\int_{\epsilon_n}^{|A_n|} |f_n(\tau(t))|^p W(t)\,dt\right]^{-p}$$

$$\times \left[\int_{\epsilon_n}^{|A_n|} |f(\tau(t))|\,|f_n(\tau(t))|^{p-1} W(t)\,dt\right]^p$$

$$\leqslant \left[\sum \int_{\epsilon_n}^{|A_n|} |f(\tau(t))|^p W(t)\,dt\right]\left[\int_{\epsilon_n}^{|A_n|} |f_n(\tau(t))|^p W(t)\,dt\right]^{-1}$$

(by Hölder's inequality)

$$\leqslant (1-\epsilon)^{-p} \quad \left[\int_0^1 |f(\tau(t))|^p W(t)\,dt\right]$$

$$= (1-\epsilon)^{-p} \|f\|^p.$$

Thus $F_n \in \Lambda(W, p)^*$ and by (5.2) $Pf = \sum F_n(f) f_n$ is a projection from $\Lambda(W, p)$ onto $[f_n]$ with $\| P \| \leqslant (1-\epsilon)^{-1}$.

Remark 5.3. Suppose (f_n) is a disjointly supported sequence of unit vectors in $\Lambda(W, p)$, $A_n = \operatorname{supp} f_n$, and $\tau_n : [0, |A_n|] \to A_n$ is measure preserving with

$$\int_0^{|A_n|} |f_n(\tau(t))|^p W(t)\,dt = 1.$$

We can choose by induction a decreasing sequence (ϵ_n) of positive numbers and an increasing sequence $1 = k_1 < k_2 <$ of positive integers to satisfy for each $f \in [f_i]_{i=1}^{k_n}$, there is a measure preserving mapping τ:

$$[0, \textstyle\sum_{i=1}^{k_n} |A_i|] \to \bigcup_{i=1}^{k_n} A_i \text{ so that} \int_{\epsilon_n}^1 |f(\tau(t))|^p W(t)\,dt \geqslant (1-\epsilon)^p \|f\|^p \quad (5.4)$$

$$\epsilon_n > \sum_{i=k_{n+1}}^{\infty} |A_i|. \quad (5.5)$$

Let $E_n = [f_i]_{i=k_n}^{k_{n+1}-1}$. The proof of Theorem 5.1 shows that for $g_n \in E_{2n}$ (or $g_n \in E_{2n-1}$),

$$(1-\epsilon)\left(\sum \|g_n\|^p\right)^{1/p} \leqslant \left\|\sum g_n\right\| \leqslant \left(\sum \|g_n\|^p\right)^{1/p}.$$

Thus (E_{2n}) and (E_{2n-1}) are both l_p decompositions. But since (E_n) is an unconditional decomposition, we have that (E_n) is an l_p decomposition. That is, $[f_n]$ is isomorphic to $(\sum E_n)_{l_p}$.

Remark 5.6. It follows from the fact that $W(1) > 0$ that $\Lambda(W, p) \subseteq L_p[0, 1]$ and the injection is continuous. The proofs of Theorems 4.1 and 5.1 thus show that a subspace of $\Lambda(W, p)$ either embeds isomorphically into $L_p[0, 1]$ or contains a complemented subspace isomorphic to l_p.

Remark 5.7. If $p \geqslant 2$, it follows from Remark 5.6 that $\Lambda(W, p) \subseteq L_2[0, 1]$ and the injection is continuous. Thus the technique of Kadec–Pelczynski [18] and Theorem 5.1 yield that if X is a subspace of $\Lambda(W, p)$ $(p \geqslant 2)$ then either X is isomorphic to a Hilbert space and X is complemented in $\Lambda(W, p)$, or X contains a subspace $1 + \epsilon$-isometric to l_p and $1 + \epsilon$-complemented in $\Lambda(W, p)$. It seems to us that the technique in [16] can be used to show that if X is a subspace of $\Lambda(W, p)$ $(p \geqslant 2)$ and no subspace of X is isomorphic to l_2, then X is isomorphic to a subspace of $(\sum E_n)_{l_p}$ for some sequence (E_n) of finite dimensional subspaces of $\Lambda(W, p)$. However, we did not check this out.

The next result is that a complemented subspace of $\Lambda(W, p)$ for $1 \leqslant p < 2$ is either isomorphic to a Hilbert space or contains a complemented subspace isomorphic to l_p. In the case of L_p with $1 < p < 2$, Kadec and Pelczynski [18] pointed out that this result follows by duality from their investigation of L_r with $2 < r < \infty$. Since $\Lambda(W, p)^*$ is not necessarily a Lorentz function space, a different approach is required here.

We use a simple fact concerning "diagonals" of operators. The fact can be proved as in [22, p. 23], or a proof using Rademacher functions can be given along the lines of the proof of Lemma 2 in [16].

Fact 5.8. Suppose X has an unconditional basis (e_n), $T: X \to \Lambda(W, p)$ is an operator, and (E_n) is a pairwise disjoint sequence of measurable subsets of $[0, 1]$. Let

$$D \sum \alpha_n e_n = \sum \alpha_n T e_n \chi_{E_n}.$$

Then D maps X into $\Lambda(W, p)$ and $\| D \| \leqslant \| T \| \, U(e_n)$.

THEOREM 5.9. *Suppose X is a complemented subspace of $\Lambda(W, p)$ $(1 \leqslant p < \infty)$. If X is not isomorphic to a Hilbert space, then X has a complemented subspace which is isomorphic to l_p.*

Proof. First assume $p > 1$ and let P be a projection from $\Lambda(W, p)$ onto X. If the conclusion is false, then by Theorems 5.1, 4.1, Remark 4.3, and the reflexivity of $\Lambda(W, p)$ there is a sequence (F_n) of norm one functionals in $P^*\Lambda(W, p)^*$ and a pairwise disjoint sequence A_n of measurable subsets of

[0, 1] which satisfy $\|F_n - \chi_{A_n} F_n\| < \epsilon$ ($\epsilon > 0$ is to be specified later). We identify $\Lambda(W, p)^*$ with a space of measurable functions on [0, 1], so that $G(f) = \int_0^1 G(t) f(t)\, dt$ for $G \in \Lambda(W, p)^*$ and $f \in \Lambda(W, p)$. Pick $g_n \in \Lambda(W, p)$ with $\operatorname{supp} g_n \subseteq A_n$, $\|g_n\| = 1$, and $(\chi_{A_n} F_n)\, g_n > 1 - \epsilon$.

In view of Theorem 5.1, we can assume by passing to a subsequence of (g_n) that (g_n) is equivalent to the unit vector basis of l_p. Thus $g_n \to 0$ weakly, hence $Pg_n \to 0$ weakly. But observe that $\inf \| Pg_n \| > 0$. Indeed, $\| Pg_n \| \geqslant \chi_{A_n} F_n(Pg_n) = P^*(\chi_{A_n} F_n)(g_n)$ and $\| P^* \chi_{A_n} F_n - F_n \| = \| P^* \chi_{A_n} F_n - P^* F_n \| \leqslant \| P^* \| \| \chi_{A_n} F_n - F_n \|$. So $\| P^* \chi_{A_n} F_n - \chi_{A_n} F_n \| \leqslant \epsilon(\| P^* \| + 1)$. Hence $|(P^* \chi_{A_n} F_n - \chi_{A_n} F_n)\, g_n| < \epsilon(\| P \| + 1)$ whence $(P^* \chi_{A_n} F_n)\, g_n > (1 - \epsilon) - \epsilon(\| P \| + 1)$, which is positive if $\epsilon > 0$ is sufficiently small.

By Remark 5.6, we can assume that the $L_p[0, 1]$ norm is equivalent to the $\Lambda(W, p)$ norm on X and hence X is isomorphic to a subspace of $L_p[0, 1]$. Since $L_p[0, 1]$ has an unconditional basis (the Haar functions), some subsequence of (Pg_n) is unconditionally basic by the results of [3]. So for simplicity of notation, assume that (Pg_n) is unconditionally basic. By Fact 5.8, the function D: $[Pg_n] \to \Lambda(W, p)$ defined by $D(\sum \alpha_n Pg_n) = \sum \alpha_n (Pg_n)\, \chi_{A_n}$ is a bounded operator. Now $\|(Pg_n)\, \chi_{A_n}\| \geqslant F_n[(Pg_n)\, \chi_{A_n}] = P^*(\chi_{A_n} F_n)\, g_n > (1 - \epsilon) - \epsilon(\| P \| + 1)$, so by Theorem 5.1 we can assume by passing to a subsequence of (g_n) that $((Pg_n)\, \chi_{A_n})$ is equivalent to the unit vector basis of l_p.

The arguments in the preceding paragraph show that the linear extension of the map $g_n \to (Pg_n)\, \chi_{A_n}$ is essentially the identity on l_p, hence the linear extension of the map $g_n \to Pg_n$ is an isomorphism, whence (Pg_n) is equivalent to the unit vector basis of l_p. Recalling that X is closed in $L_p[0, 1]$ we have that $[Pg_n]$ is closed in $L_p[0, 1]$ and (Pg_n) is, in $L_p[0, 1]$, equivalent to the unit vector basis of l_p. Thus by a result of Enflo and Rosenthal [10], there is a subsequence (Pg_{n_i}) of (Pg_n) and a projection Q from $L_p[0, 1]$ onto $[Pg_{n_i}]$. Of course, the restriction of Q to X is a projection of X onto a subspace of X isomorphic to l_p. This completes the proof of the case when $p > 1$.

Assume now that $p = 1$. In this case the only problem occurs when X is closed in $L_1[0, 1]$. If X is not reflexive, then X contains a complemented subspace isomorphic to l_1 by a theorem of Kadec–Pelczynski [18], so we assume that X is reflexive. Since $\Lambda(W, 1)^*$ is not σ-order continuous, we cannot apply Theorem 4.1 and Remark 4.3 directly. However, we can get around this problem by using an argument from [30]: Since X is reflexive, if $F \in \Lambda(W, p)^*$ and (A_n) is a disjoint sequence of measurable subsets of [0, 1], then $\| \chi_{A_n} F_{|X} \| \to 0$. Indeed, if $\| \chi_{A_n} F \| \geqslant \delta > 0$ for infinitely many n (for $n \in A$, say) then $(\chi_{A_n} F)_{n \in A}$ is equivalent to the unit vector basis of c_0 because it is an unconditional basic sequence and $\| \sum_{n \in B} \chi_{A_n} F \| \leqslant \| F \|$ for all subsets B of A. The restriction operator $\sum_{n \in A} \alpha_n \chi_{A_n} F \to \sum_{n \in A} \alpha_n \chi_{A_n} F_{|X}$ is compact (as is any bounded linear operator from c_0 into a reflexive space), hence $\| \chi_{A_n} F_{|X} \| \to 0$. This fact and the proof of Theorem 4.1 yield that there is a sequence (F_n) of norm one functionals in $P^*\Lambda(W, p)^*$ and a pairwise disjoint sequence (A_n) of measurable subsets of $[0, 1]$ so that $\inf \| \chi_{A_n} F_{n|X} \| > 0$. One can choose a sequence f_n of norm one elements of X so that $\liminf F_n(\chi_{A_n} f_n) = \liminf (\chi_{A_n} F_n) f_n > 0$. We can assume, by passing to a subsequence of (f_n), that $f_n \to f$ weakly. Now $(\chi_{A_n} F_n)_{|X} \to 0$ weak* since (A_n) is pairwise disjoint and X is reflexive, so $(\chi_{A_n} F_n) f \to 0$ and by replacing (f_n) with a subsequence of $(f_n - f)$, we may assume that $f_n \to 0$ weakly. Since X is isomorphic to a subspace of $L_1[0, 1]$ and is reflexive, X is isomorphic to a subspace of $L_r[0, 1]$ for some $1 < r \leqslant 2$ by a theorem of Rosenthal's [28]. Thus, as in the proof of the $p > 1$ case, we can assume that (f_n) is unconditionally basic. Hence by Fact 5.8, the map $\sum \alpha_n f_n \to \sum \alpha_n \chi_{A_n} f_n$ is a bounded linear operator from $[f_n]$ to $[\chi_{A_n} f_n]$. Now $\liminf \| \chi_{A_n} f_n \| \geqslant \liminf F_n(\chi_{A_n} f_n) > 0$, so by Theorem 5.1, some subsequence of $(\chi_{A_n} f_n)$ is equivalent to the unit vector basis of l_1. Since $[f_n]$ is reflexive, this is a contradiction. This completes the proof.

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